Test #2 - Solutions

[5] #1. (a) Evaluate $\cosh \frac{\pi i}{2}$

by definition $\cosh z = \dfrac{e^{z}+e^{-z}}{2}$

and $\cosh iz = \dfrac{e^{iz}+e^{-iz}}{2} = \cos z$

so $\cosh \frac{\pi i}{2} = \cos \frac{\pi}{2} = 0$

(b) Solve for z: $\sin z = \frac{i}{2}$

By definition $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = i$

multiply both sides by $2i$

$$e^{iz} - e^{-iz} = -1$$

or

$$(e^{iz} - e^{-iz} + 1 = 0) \times e^{iz}$$

$$e^{2iz} + e^{iz} - 1 = 0$$

Set $w = e^{iz}$ $\Rightarrow$ $w^2 + w - 1 = 0$

use the quadratic equation to find the roots

$$w = \frac{-1 \pm \sqrt{1+4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Next, we solve for $e^{iz} = \frac{-1 \pm \sqrt{5}}{2}$

Take log of both sides

$$-i\left(iz = \operatorname{Log}\left|-\tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}\right| + i\operatorname{Arg}\left(-\tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}\right) + 2k\pi i \qquad k \in \mathbb{Z}\right)$$

$$z = -i\operatorname{Log}\left|-\tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}\right| - \operatorname{Arg}\left(-\tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}\right) + 2k\pi$$

since $-1+\sqrt{5} > 0$ and $-1-\sqrt{5} < 0$ are both real

$$\operatorname{Arg}\left(-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right) = 0 \quad \text{and} \quad \operatorname{Arg}\left(-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}\right) = \pi$$

The roots are

$$z_+ = -i\operatorname{Log}\left|-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right| + 2k\pi \qquad k \in \mathbb{Z}$$

$$z_- = -i\operatorname{Log}\left|-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}\right| + (2k+1)\pi$$

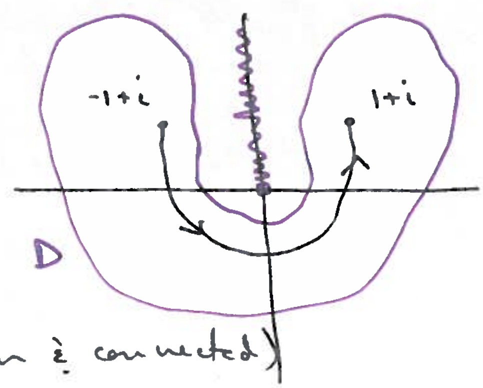
there are ∞-many values.

[4] # 2. (a) Fundamental theorem of Calculus for contours

Let Γ be any contour in a domain D with end points z_0 and z_1. If $f: D \to \mathbb{C}$ is continuous and has an antiderivative $F(z)$ in D, then

$$\int_\Gamma f(z)\,dz = F(z_1) - F(z_0).$$

(b) Compute $\displaystyle\int_\gamma \frac{1}{z}\,dz$



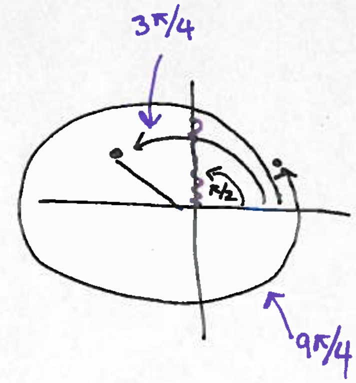
$f(z)$ is continuous in D (open & connected) and has an antiderivative in D

$$\frac{d}{dz}\left(\mathcal{L}_{\frac{\pi}{2}} z\right) = \frac{1}{z} \qquad \text{where}$$

$$\mathcal{L}_{\frac{\pi}{2}}(z) = \operatorname{Log}|z| + i\arg_{\frac{\pi}{2}} z$$

and $\quad \frac{\pi}{2} < \arg_{\frac{\pi}{2}} z \le \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$

By the F.T.C for contours



$$\int_\gamma \frac{1}{z}\,dz = \mathcal{L}_{\frac{\pi}{2}}(z)\Big|_{-1+i}^{1+i}$$

$$= \mathcal{L}_{\frac{\pi}{2}}(1+i) - \mathcal{L}_{\frac{\pi}{2}}(-1+i)$$

$$= \operatorname{Log}\sqrt{2} + i\frac{9\pi}{4} - \operatorname{Log}\sqrt{2} - i\frac{3\pi}{4}$$

$$= i\,\frac{6\pi}{4} = i\,\frac{3\pi}{2}$$

5] #3. (a) Cauchy Integral Theorem

If f is analytic in a simply connected domain D and Γ is any loop in D, then

$$\oint_\Gamma f(z)\,dz = 0$$

(b) Deformation Invariance theorem

Let f be an analytic function in a domain D containing the loops γ_0 and γ_1. If these loops can be continuously deformed into one another in D then

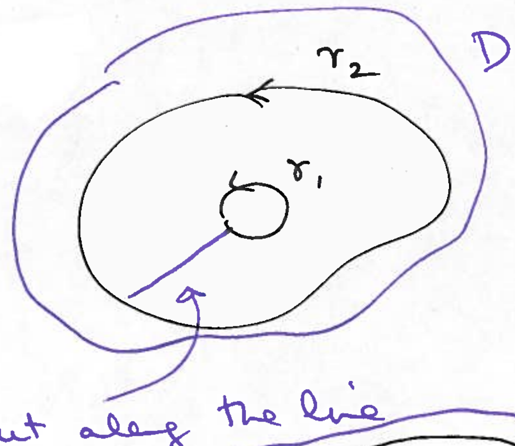
$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$$

#3(c) Show (a) ⇒ (b)

Assume f is analytic in a domain D and that γ_1 and γ_2 are two loops in D that can be continuously deformed into one another.

we want to show that $\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$

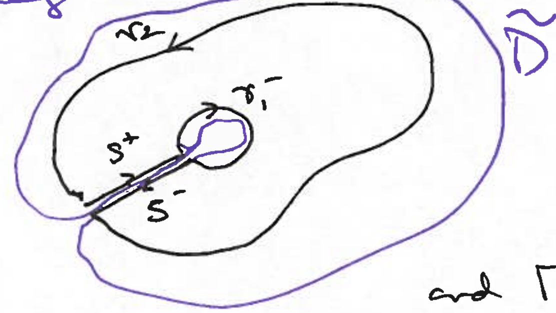
Let



Consider The contour

$$\Gamma = \{\gamma_2, S, -\gamma_1, -S\}$$

obtained as follows.



the $\tilde{D}$ is simply connected, f is analytic in $\tilde{D} \subset D$ and Γ is a loop in D.
so by Cauchy's Integral Theorem

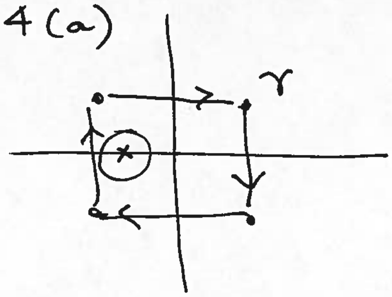
$$\int_{\Gamma} f(z)\,dz = 0 = \int_{\gamma_2} f(z)\,dz + \int_{S} f(z)\,dz + \int_{\gamma_1^-} f(z)\,dz + \int_{S^-} f(z)\,dz$$

since $\int_{S^-} f(z)\,dz = -\int_{S} f(z)\,dz$, and $\int_{\gamma_1^-} f(z)\,dz = -\int_{\gamma_1} f(z)\,dz$

we have

$$0 = \int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz \iff \int_{\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

[8] 4 (a)



note that γ is not positively oriented.

but

$$\int_\gamma \frac{\sin z}{4z+\pi}\,dz = -\int_{\gamma^+} \frac{\sin z}{4z+\pi}\,dz$$

where γ^+ is the same loop with positive orientation

also – $\sin z$ is entire / analytic on all of $\mathbb{C}$

$\frac{1}{4z+\pi} = \frac{1}{4(z+\frac{\pi}{4})}$ has a singularity at $z=-\frac{\pi}{4}$ inside the loop

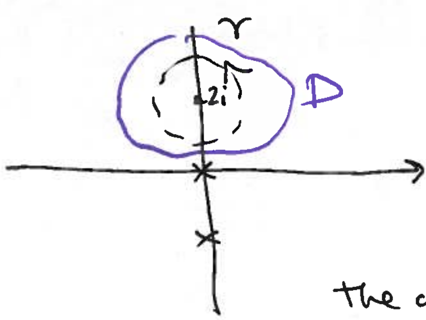
We use Cauchy Integral formula. with $f(z) = \sin z$
D any open set containing γ.

$$\frac{1}{4}\oint_{\gamma^+} \frac{\sin z}{z+\frac{\pi}{4}}\,dz = \frac{1}{4}\,2\pi i \sin\frac{-\pi}{4}$$

$$= \frac{\pi i}{2}\sin\left(-\frac{\pi}{4}\right) = -\frac{\pi i}{2\sqrt{2}}$$

So $\int_\gamma \frac{\sin z}{4z+\pi}\,dz = \frac{-\pi i}{2\sqrt{2}}$

(b)



Consider

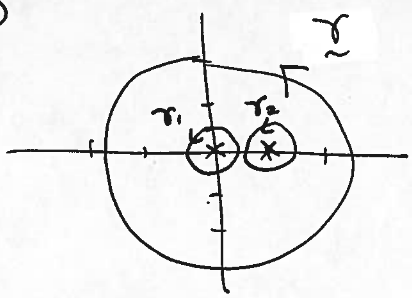
$$\oint_\gamma \frac{z+i}{z^3+2z^2}\,dz = \oint_\gamma \frac{z+i}{z^2(z+2)}\,dz$$

the only singularities occur at $z=0$ & $z=-2$
both are outside D - simply connected

So $f(z) = \frac{z+i}{z^3+2z^2}$ is analytic in D,

by Cauchy's integral theorem $\oint_\gamma \frac{z+i}{z^3+2z^2}\,dz = 0$

4 (c)



$$\oint_{\gamma} \frac{2-z}{z^2-z}\,dz$$

note that $\dfrac{2-z}{z^2-z} = \dfrac{2-z}{z(z-1)}$ is undefined at $z=0$ and $z=1$ and both points are in the interior of γ

since $f(z)$ is analytic away from the singularities we can use deformation invariance to obtain

$$\oint_{\gamma} \frac{2-z}{z^2-z}\,dz = \int_{\gamma_1} \frac{2-z}{z^2-z}\,dz + \int_{\gamma_2} \frac{2-z}{z^2-z}\,dz$$

Method 1: Use Cauchy Integral formulas.

$$\int_{\gamma_1} \frac{2-z/z-1}{z}\,dz = 2\pi i\left(\frac{2-z}{z-1}\right)\Bigg|_{z=0} = -4\pi i$$

and $$\int_{\gamma_2} \frac{2-z/z}{z-1}\,dz = 2\pi i\left(\frac{2-z}{z}\right)\Bigg|_{z=1} = 2\pi i$$

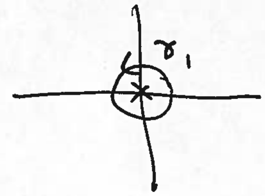
This works because in each case. the function in the numerator is analytic in the domain that contains the loops γ_i and γ but not the singular part. So $\oint_{\gamma} \frac{2-z}{z^2-z}\,dz = -2\pi i$

Method 2: Use partial fraction

$$\frac{2-z}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1)+Bz}{z(z-1)} = \frac{(A+B)z-A}{z(z-1)}$$

set $A+B=-1$
and $A=-2$ $\Rightarrow$ $B=1$

So $$\frac{2-z}{z(z-1)} = \frac{-2}{z} + \frac{1}{z-1}$$

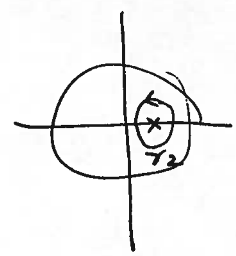


then
$$\oint_{\gamma_1} \frac{2-z}{z(z-1)}\,dz = \underbrace{\oint_{\gamma_1} -\frac{2}{z}\,dz}_{=\,-2(2\pi i)} + \underbrace{\oint_{\gamma_1} \frac{1}{z-1}\,dz}_{=\,0} = -4\pi i$$

$= 0$ by Cauchy Integral theorem

$-2(2\pi i)$ by the $2\pi i$ theorem

$$\oint_{\gamma_2} \frac{2-z}{z(z-1)}\,dz = \underbrace{\oint_{\gamma_2} -\frac{2}{z}\,dz}_{=\,0} + \underbrace{\oint_{\gamma_2} \frac{1}{z-1}\,dz}_{=\,2\pi i}$$



0 by Cauchy's Integral theorem

and
$$\oint_{\gamma} \frac{2-z}{z(z-1)}\,dz = -4\pi i + 2\pi i = -2\pi i$$

[3] #5. Compute $\oint_{\gamma_R} \frac{dz}{(z-z_0)^n}$.

we parametrize

$$\gamma_R : z(t) = z_0 + Re^{it} \qquad 0 \le t \le 2\pi$$

$$z'(t) = iRe^{it}$$

$$f(z) = \frac{1}{(z-z_0)^n} = \frac{1}{(z_0 + Re^{it} - z_0)^n} = \frac{1}{R^n e^{int}}$$

for $n=1$

$$\oint_{\gamma_R} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{1}{Re^{it}} \cdot iRe^{it}\,dt = it\Big|_0^{2\pi} = 2\pi i$$

for $n \neq 1$

$$\oint_{\gamma_R} \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{1}{R^n e^{int}} \cdot iRe^{it}\,dt$$

$$= \frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)t}\,dt$$

$$= \frac{i\,(i(1-n))}{R^{n-1}}\, e^{i(1-n)t}\Big|_0^{2\pi}$$

$$= \frac{i(i(1-n))}{R^{n-1}} \Big(\underbrace{e^{i(1-n)2\pi}}_{\substack{\text{multiple of } 2\pi \\ = \\ 1}} - \underbrace{e^0}_{\substack{= \\ 1}} \Big) = 0$$